Harmonic fields on the extended projective disc and a problem in optics

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Abstract

The Hodge equations for 1-forms are studied on Beltrami's projective disc model for hyperbolic space. Ideal points lying beyond projective infinity arise naturally in both the geometric and analytic arguments. An existence theorem for weakly harmonic 1-fields, changing type on the unit circle, is derived under Dirichlet conditions imposed on the non-characteristic portion of the boundary. A similar system arises in the analysis of wave motion near a caustic. A class of elliptic-hyperbolic boundary-value problems is formulated for those equations as well. For both classes of boundary-value problems, an arbitrarily small lower-order perturbation of the equations is shown to yield solutions which are strong in the sense of Friedrichs. MSC2000: 35M10, 58J32, 53A20, 78A05

1 Introduction

The projective disc was introduced by Beltrami[3] in 1868. His construction was an early example of a Euclidean model for a non-Euclidean space, in this case, a space having curvature equal to -1. The projective disc has the striking property that even points infinitely distant from the origin are enclosed by the Euclidean unit circle centered at the origin of $\mathbb{R}^2$. This implies

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the possibility of points in projective space which lie beyond the curve at infinity. It is known that such *ideal points* arise naturally in the process of constructing normal and translated lines for chords of the projective disc. In this sense ideal points may be said to be intrinsic to the model, rather than only a theoretical possibility allowed by the model. We call the union of the conventional projective disc $\mathbb{P}^2$ and its ideal points the *extended projective disc.*

Hua[9] considered a second-order partial differential equation for scalar functions on the extended projective disc. He proved the existence of solutions to certain boundary-value problems of *Tricomi type,* in which data are given on characteristic curves, which represent trajectories of generalized wavefronts. Hua's work was extended to other problems of Tricomi type by Ji and Chen.[10,11] The existence of a class of weak solutions to the Hodge equations for harmonic 1-fields on extended $\mathbb{P}^2$, with data prescribed only on the non-characteristic part of the boundary, was proven in Ref. 23. Locally, the Hodge equations reduce in the smooth scalar case to the equation studied by Hua.

This communication provides a geometric and analytic context for such results (Sec. 1). In addition, we prove an existence theorem for weakly harmonic 1-fields which includes the results of Ref. 23 as a special case (Sec. 2.1), and consider a similar system that arises in optics (Secs. 3.1, 3.2). Boundary-value problems are formulated for both systems, in which the boundary contains points in both the elliptic and hyperbolic regions of the equations. These problems are shown in Secs. 2.2 and 3.3 to be an arbitrarily small, lower-order perturbation away from problems possessing a unique, strong solution.

Because both scalar equations and systems are discussed, we distinguish a vector-valued solution by writing it in boldface. However, for typographic simplicity, coefficient matrices and operators are not written in boldface.

1.1 A geometric classification of linear second-order operators

The highest-order terms of any linear second-order partial differential equation on a domain $\Omega \subset \mathbb{R}^2$ can be written in the form

$$Lu = \alpha\,(x,y)\,u_{xx} + 2\beta(x,y)u_{xy} + \gamma(x,y)u_{yy},$$

where (x, y) are coordinates on Ω and α, β, and γ are given functions. (A subscripted variable denotes partial differentiation in the direction of the variable.)

If the discriminant
$$\Delta(x, y) = \alpha\gamma - \beta^2$$
is positive, then the equation associated with the operator L is said to be of *elliptic type*. The simplest example is Laplace's equation, for which $\alpha = \gamma = 1$ and $\beta = 0$. If the discriminant is negative, then the equation associated with the operator L is said to be of *hyperbolic type*. The simplest example is the normalized wave equation, for which $\alpha = 1$, $\gamma = -1$, $\beta = 0$; other forms are $\alpha = -1$, $\gamma = 1$, $\beta = 0$, or $\alpha = \gamma = 0$, $\beta = 1$. If $\Delta = 0$, then the equation associated with the operator L is said to be of *parabolic type*; examples are equations which model diffusion. If the discriminant is positive on part of Ω and negative elsewhere on Ω, then the equation associated with the operator L is said to be of *mixed elliptic-hyperbolic type*. A simple example of an elliptic-hyperbolic equation is the Lavrent'ev-Bitsadze equation, for which $\alpha = sgn(y)$, $\beta = 0$, and $\gamma = 1$.

If we take Ω to be a smooth but curved surface, then we may not be able to cover Ω by a single system of Cartesian coordinates. However, we can always introduce Cartesian coordinates (x^1, x^2) *locally* on any smooth surface, in the neighborhood of a point on the surface. In terms of such coordinates, the distance element ds on Ω can be written in the form

$$ds^2 = \sum_{i=1}^{2}\sum_{j=1}^{2} g_{ij}(x^1, x^2)dx^i dx^j,$$

where g_{ij} is a symmetric 2×2 matrix, the *metric tensor* on Ω. (In the sequel we will understand repeated indices to have been summed from 1 to $\dim(\Omega)$ without writing out the summation notation each time.) A natural differential operator on functions u defined on such a space is the *Laplace-Beltrami operator*
$$Lu = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\left(g^{ij}\sqrt{|g|}\frac{\partial u}{\partial x^j}\right),$$
where g^{ij} is the inverse of the matrix g_{ij} and g is its determinant.

Laplace's equation can be associated to the Laplace-Beltrami operator on the Euclidean metric for which g_{ij} is the identity matrix. The wave equation for $\beta = 0$ can be associated with the Laplace-Beltrami operator

on the 2-dimensional Minkowski metric $g_{11} = 1$, $g_{22} = -1$, $g_{12} = g_{21} = 0$. The Lavrent'ev-Bitsadze equation can be associated to the Laplace-Beltrami operator on a metric which is Euclidean above the x-axis and Minkowskian below the x-axis.

In this classification, the type of a linear second-order equation is not a function of the associated linear operator at all; that operator is always the Laplace-Beltrami operator. Rather, the type of the equation is a feature of the metric tensor on an underlying surface. A Riemannian metric, in which the distance between distinct points of Ω is always positive, corresponds to an elliptic equation, whereas a pseudoriemannian metric, for which the distance between distinct points may be zero, corresponds to a hyperbolic equation. The Laplace-Beltrami operator on a surface for which the metric is Riemannian on part of a surface and pseudoriemannian elsewhere will be of mixed elliptic-hyperbolic type. However, any *sonic* – or *parabolic* – curve on which the change of type occurs will necessarily represent a singularity of the metric tensor, as the determinant g will vanish along that curve. (The term *sonic curve* is borrowed from compressible fluid dynamics, in which the equations for the velocity field of a steady ideal flow change from elliptic to hyperbolic type at the speed of sound. The underlying pseudoriemannian metric in that case is called the *flow metric.*[4])

One definition of the *signature* of a metric is the sign of the diagonal entries of the metric tensor. Any change in the signature which results in a change in sign of the determinant g will change the Laplace-Beltrami operator on the metric from elliptic to hyperbolic type. The Laplace-Beltrami operator on surface metrics for which such a change occurs along a smooth curve will correspond to planar elliptic-hyperbolic operators in local coordinates.

If we consider the distance element

$$ds_L^2 \equiv \alpha\,(x,y)\,dy^2 - 2\beta\,(x,y)\,dxdy + \gamma\,(x,y)\,dx^2,$$

then *null geodesics* on the corresponding surface are solutions of the ordinary differential equation

$$ds_L^2 = 0.$$

The graphs of these solutions are called *characteristic curves* of the equation $Lu = 0$. Hyperbolic operators, which are associated with wave propagation, always have real-valued *characteristics,* or null geodesics.

In determining the qualitative behavior of solutions to partial differential equations we often ignore lower-order terms, but this neglect is only justified

when considering purely second-order properties such as the nature of the sonic curve. The importance to this paper of lower-order terms is related to the fact that the Laplace-Beltrami equations on the extended projective disc are not of *real principal type* in the sense of Ref. 6; see Ref. 27 for an accessible discussion of scalar elliptic-hyperbolic operators of real principal type and their properties.

1.2 The geometry and analysis of ideal points

Here we review basic properties of Laplace-Beltrami equations on Beltrami's hyperbolic metric on the projective disc:

$$ds^2 = \frac{(1-y^2)\,dx^2 + 2xydxdy + (1-x^2)\,dy^2}{(1-x^2-y^2)^2}$$

(see , *e.g.,* Ref. 32, Vol. I, Sec. 65 and Vol. II, Sec. 138, for a derivation). In this metric the unit circle is the *absolute*: the locus of points at infinity.

The existence of points lying beyond the curve at infinity on the projective disc is natural from a geometric point of view. For example, choose a point p in the interior of the projective disc and draw a vertical line ℓ_v through it. A *hyperbolic line* in the Beltrami metric is any open chord of the unit circle, so ℓ_v is a hyperbolic line plus two points at infinity and an *ideal extension* to points outside the unit circle. Denote by $F(p)$ the family of hyperbolic lines created by rotating ℓ_v about p. Move p along the horizontal line ℓ_h through p, and consider the affect of this motion on the family $F(p)$. As p passes through the boundary of the unit circle κ into the $\mathbb{R}^2$-complement of κ, the family of hyperbolic rotations becomes a family of hyperbolic translations. For this reason, hyperbolic translations inside the unit disc can be interpreted as rotations about a point in $\mathbb{R}^2$ lying beyond the unit disc.

As another example, consider that the *pole* of a hyperbolic line ℓ is the intersection of those two tangents to the unit circle which intersect ℓ at the two points of its contact with the unit circle. (We call these the *polar lines* of ℓ.) Thus any two hyperbolic lines ℓ_1 and ℓ_2 are orthogonal if and only if the pole of ℓ_2 lies on the ideal extension of ℓ_1 and *vice-versa*.

These and other geometric constructions on extended $\mathbb{P}^2$ are described in more detail in Chapter 4 of Ref. 28.

In order to see that ideal points also arise naturally in analysis, consider the Laplace-Beltrami operator on the projective disc with Beltrami's metric.

We have

$$L\left[u\right]=\left(1-x^{2}-y^{2}\right)\left[\left(1-x^{2}\right)u_{xx}-2xyu_{xy}+\left(1-y^{2}\right)u_{yy}\right.$$

$$\left.+lower\ order\ terms\right].$$

The characteristics of the equation $L[u]=0$ satisfy the ordinary differential equation

$$\left(1-y^{2}\right)dx^{2}+2xydxdy+(1-x^{2})dy^{2}=0. \tag{1}$$

This equation has solutions

$$x\cos\theta+y\sin\theta=1, \tag{2}$$

where, as is conventional, we take θ to be the angle between the radial vector and the positive x-axis. Solutions of eq. (2) correspond geometrically to the family of tangent lines to the unit circle centered at the origin of $\mathbb{R}^2$.

Thus the characteristic lines always include ideal points and wave propagation can only occur on regions composed of such points.

The Laplace-Beltrami equations on extended $\mathbb{P}^2$ come with a natural gauge theory in the following sense: The characteristic equation is obviously invariant under the projective group. So although the equations in the form in which we study them change type on the unit circle in $\mathbb{R}^2$, they are projectively equivalent to a system which changes type on any conic section. Note that whereas classical gauge theories are invariant under groups of Euclidean motions, which are inertial transformations, this kind of gauge invariance is with respect to a group of non-Euclidean motions, which are non-inertial. Also, the gauge theories which are familiar from particle physics act "upstairs" on a fiber bundle of physical states. The transformation group under which the Laplace-Beltrami equations are invariant acts "downstairs" on the underlying metric, in the manner of the gauge group of general relativity. Indeed, analysis of wave motion on extended $\mathbb{P}^2$ has certain similarities to the analysis of wave motion in the vicinity of a light cone (*c.f.* Ref. 30). The time-like and space-like regions are inverted, and characteristic lines for the Laplace-Beltrami equation are analogous to the paths of photons.

2 Harmonic 1-fields on the extended projective disc

We can solve, instead of the Laplace-Beltrami equation, a system of two first-order equations of the form

$$|g|^{-1/2}\,\partial_i\left(g^{ij}\sqrt{|g|}u_j\right)=0, \tag{3}$$

$$\frac{1}{2}\left(\partial_i u_j-\partial_j u_i\right)=0, \tag{4}$$

where $u_i = u_i(x^1, x^2)$, $i = 1, 2$. As in the second-order equation, g_{ij} is a metric tensor on the underlying surface. Solutions $\mathbf{u} = (u_1, u_2)$ of this first-order system are (locally) *harmonic 1-fields.* Notice that if the scalar function $\varphi\left(x^1, x^2\right)$ satisfies $\varphi_{x^1} = u_1$ and $\varphi_{x^2} = u_2$, then φ satisfies the Laplace-Beltrami equations. But there are solutions φ of the Laplace-Beltrami system for which the pair $(\varphi_{x^1}, \varphi_{x^2})$ is not a harmonic 1-field.

Consider a system of first-order equations on $\mathbb{R}^2$ having the form

$$L\mathbf{u}=\mathbf{f}, \tag{5}$$

where

$$L=\left(L_1,L_2\right),\ \mathbf{f}=\left(f_1,f_2\right),$$

$$\mathbf{u}=\left(u_1\left(x,y\right),u_2\left(x,y\right)\right),\ (x,y)\in\Omega\subset\subset\mathbb{R}^2.$$

Let $\mathbf{u}$ satisfy (5) with

$$(L\mathbf{u})_1=\left[\left(1-x^2\right)u_1\right]_x-2xyu_{1y}+\left[\left(1-y^2\right)u_2\right]_y+k_1xu_1+k_2yu_2, \tag{6}$$

and

$$(L\mathbf{u})_2=\left(1-y^2\right)\left(u_{1y}-u_{2x}\right)+k_3xu_1+k_4yu_2, \tag{7}$$

where Ω is chosen so that $y^2 \neq 1$ there. Here k_1, k_2, k_3 and k_4 are constants representing lower-order coefficients. In this section we consider three particular distributions of lower-order terms, studied in Ref. 23:

Case 1: $k_1 = k_2 = -2$, $k_3 = k_4 = 0$. The domain of eqs. (5)-(7) in this case will be called Ω_1.

Case 2: $k_1 = -2$, $k_2 = k_3 = 0$, $k_4 = 2$. The domain of eqs. (5)-(7) in this case will be called Ω_2.

Case 3: $k_1 = k_2 = k_3 = k_4 = 0$. The domain of eqs. (5)-(7) in this case will be called Ω_3. This case corresponds to eqs. (3), (4).

The union of the domains Ω_1, Ω_2, and Ω_3 will be called Ω.

A system of first-order equations can also be said to be of elliptic or hyperbolic type, and thus may change type along a singular curve. See, *e.g.,* Ref. 5, Ch. III.2. The higher-order terms of the preceding system can be written in the form $A^1\mathbf{u}_x + A^2\mathbf{u}_y$, where

$$A^1 = \begin{bmatrix} 1-x^2 & 0 \\ 0 & -\left(1-y^2\right) \end{bmatrix} \tag{8}$$

and

$$A^2 = \begin{bmatrix} -2xy & 1-y^2 \\ 1-y^2 & 0 \end{bmatrix}. \tag{9}$$

If $y^2 \neq 1$, the characteristic equation

$$\left|A^1 - \lambda A^2\right| = -\left(1-y^2\right)\left[\left(1-y^2\right)\lambda^2 + 2xy\lambda + \left(1-x^2\right)\right]$$

possesses two real roots λ_1, λ_2 on Ω precisely when $x^2 + y^2 > 1$. Thus the system is elliptic in the intersection of Ω with the open unit disc centered at $(0,0)$ and hyperbolic in the intersection of Ω with the complement of the closure of this disc. The boundary of the unit disc, along which this change in type occurs, is the line at infinity on the projective disc and a line singularity of the tensor g_{ij}.

Denote by Ω a region of the plane for which part of the boundary $\partial\Omega$ consists of a family of curves Γ composed of points satisfying eq. (1) and the remainder $C = \partial\Omega\backslash\Gamma$ of the boundary consists of points (x,y) which do not satisfy eq. (1). We seek solutions of eqs. (5)-(7) which satisfy the boundary condition

$$u_1\frac{dx}{ds} + u_2\frac{dy}{ds} = 0, \tag{10}$$

where s denotes arc length, on the non-characteristic part C of the domain boundary. Because the tangent vector $\mathbf{T}$ on C is given by

$$\mathbf{T} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{dx}\mathbf{j},$$

a geometric interpretation of this boundary condition is that the dot product of the vector $\mathbf{u} = (u_1, u_2)$ and the tangent vector to C vanishes, *i.e.*, $\mathbf{u}$ is normal to the boundary $\partial\Omega$ on the boundary section C. We call these *homogeneous Dirichlet conditions.*

2.1 Weak solutions

In Ref. 23, weak solutions to (5)-(7), (10) are shown to exist in certain weighted L^2 spaces on a class of domains. Here we extend that result to the case in which the domain is formed by the polar lines of a hyperbolic line ℓ and a smooth curve C extending between the two polar lines of ℓ. The curve C must have the property that $dy_{|C} \leq 0$ when $\partial\Omega$ is traversed in a counterclockwise direction. However, as long as this condition is met, C need not intersect the polar lines of ℓ at their points of tangency with the unit circle. Thus C may extend into both the elliptic and the hyperbolic regions.

This domain is the analogue of the "ice-cream cone"-shaped domain associated to the *Tricomi equation*[31]

$$yu_{xx} + u_{yy} = 0,$$

where in our case the curve C is the boundary of the ice-cream part and the polar lines, which are characteristics of eqs. (5)-(7), are the boundary of the cone part. The unit circle is the analogue of the x-axis, which is the sonic curve for the Tricomi equation.

We initially consider the distribution of lower-order terms in case 1 of eqs. (6), (7). Let θ lie in the interval $[0, \pi/4]$ and denote by Ω_1 the region of the first and fourth quadrants bounded by the characteristic line

$$\Gamma_1 : x\cos\theta + y\sin\theta = 1,$$

the characteristic line

$$\Gamma_2 : x\cos\theta - y\sin\theta = 1,$$

and a smooth curve C. Let C intersect the lines Γ_1, Γ_2 at two distinct points c_1, c_2, respectively. Assume that $\forall\,(x,y) \in \Omega_1$, $1/\sqrt{2} \leq x < \sqrt{2}$ and $-1/\sqrt{2} \leq y < 1/\sqrt{2}$, and that $dy \leq 0$ on C. A cusp is permitted for $\theta = \pi/4$ at the points $c_1, c_2 = (1/\sqrt{2}, \pm 1/\sqrt{2})$. Otherwise, the boundary will have piecewise continuous tangent (so that Green's Theorem can be applied to it). Note that the domain considered in Sec. 3 of Ref. 23 is equivalent to this domain in the degenerate special case $\theta = 0$.

Define U to be the vector space consisting of all pairs of measurable functions $\mathbf{u} = (u_1, u_2)$ for which the weighted L^2 norm

$$\|\mathbf{u}\|_* = \left[\int\int_{\Omega_1} \left(\left|2x^2 - 1\right| u_1^2 + \left|2y^2 - 1\right| u_2^2\right) dxdy\right]^{1/2}$$

is finite. Notice that this expression vanishes at the intersection of ℓ with its polar lines at the value $\theta = \pi/4$. Denote by W the linear space defined by pairs of functions $\mathbf{w} = (w_1, w_2)$ having continuous derivatives and satisfying:

$$w_1 dx + w_2 dy = 0 \tag{11}$$

on $\Gamma = \Gamma_1 \bigcup \Gamma_2$;

$$w_1 = 0 \tag{12}$$

on C; and

$$\int\int_{\Omega_1} \left[\left|2x^2 - 1\right|^{-1} (L^*\mathbf{w})_1^2 + \left|2y^2 - 1\right|^{-1} (L^*\mathbf{w})_2^2 \right] dxdy < \infty.$$

Here

$$(L^*\mathbf{w})_1 = \left[\left(1 - x^2\right) w_1\right]_x - 2xyw_{1y} + \left[\left(1 - y^2\right) w_2\right]_y + 2xw_1,$$

and

$$(L^*\mathbf{w})_2 = \left(1 - y^2\right)(w_{1y} - w_{2x}) + 2yw_1.$$

Define the Hilbert space H to consist of pairs of measurable functions $\mathbf{h} = (h_1, h_2)$ for which the norm

$$\|\mathbf{h}\|^* = \left[\int\int_{\Omega_1} \left(\left|2x^2 - 1\right|^{-1} h_1^2 + \left|2y^2 - 1\right|^{-1} h_2^2 \right) dxdy \right]^{1/2}$$

is finite.

We say that $\mathbf{u}$ is a *weak solution* of the system (5)-(7), (10) in case 1 on Ω_1 if $\mathbf{u} \in U$ and for every $\mathbf{w} \in W$,

$$-(\mathbf{w}, \mathbf{f}) = (L^*\mathbf{w}, \mathbf{u}),$$

where

$$(\mathbf{w}, \mathbf{f}) = \int\int_{\Omega_1} (w_1 f_1 + w_2 f_2)\, dxdy.$$

In case 2, we restrict the domain Ω_2 to lie in the fourth quadrant of the Cartesian plane. Define $\Gamma_1, \Gamma_2 \subset \Gamma$ to be characteristic lines which are tangent to the unit circle at distinct points in the fourth quadrant and which intersect at a point in the complement of the unit disc in $\mathbb{R}^2$. The curve C is defined analogously to the corresponding curve of Ω_1. In particular, $dy_{|C} \leq 0$

on Ω_2 when C is traversed in a counter-clockwise direction. Replace U by the space U' of all pairs $\mathbf{u}$ of measurable functions (u_1, u_2) for which the weighted L^2 norm

$$\|\mathbf{u}\|'_* = \left[\int\int_{\Omega_2} \left(xu_1^2 + |y|\, u_2^2\right) dxdy\right]^{1/2}$$

is finite. Replace W by the space W' defined by pairs of continuously differentiable functions $\mathbf{w} = (w_1, w_2)$ satisfying eq. (11) on Γ, eq. (12) on C, and

$$\int\int_{\Omega_2} \left[x^{-1} \left(L^*\mathbf{w}\right)_1^2 + |y|^{-1} \left(L^*\mathbf{w}\right)_2^2\right] dxdy < \infty.$$

In this case

$$\left(L^*\mathbf{w}\right)_1 = \left[\left(1 - x^2\right) w_1\right]_x - 2xyw_{1y} + \left[\left(1 - y^2\right) w_2\right]_y + 2xw_1,$$

and

$$\left(L^*\mathbf{w}\right)_2 = \left(1 - y^2\right) \left(w_{1y} - w_{2x}\right) - 2yw_2.$$

Finally, we replace H by the space H' of measurable functions $\mathbf{h} = (h_1, h_2)$ for which the norm

$$\|\mathbf{h}\|'^* = \left[\int\int_{\Omega_2} \left(x^{-1}h_1^2 + |y|^{-1} h_2^2\right) dxdy\right]^{1/2}$$

is finite.

Because k_4 is nonzero in case 2, the consistency condition (4) is violated and $\mathbf{u}$ cannot be the gradient of a scalar potential, even locally. Harmonic fields in which condition (4) is violated arise in various contexts − see Section 4 of Ref. 25 for a nonlinear example − and correspond physically to stationary fields having sources.

In case 3, we restrict the domain, Ω_3, to lie in the first quadrant. Define Γ_1, $\Gamma_2 \subset \Gamma$ to be characteristic lines which are tangent to the unit circle at distinct points in the first quadrant and which intersect at a point in the complement of the unit disc in $\mathbb{R}^2$. In this case we replace U and H by L^2. We replace W by the space of pairs of L^2 functions (w_1, w_2) which satisfy (11) on Γ and (12) on C. Note that L is self-adjoint in case 3. In addition, we fix positive numbers $\delta << 1/2$ and $\varepsilon << 1/2$ and require Ω_3 to lie in the semi-infinite rectangle

$$\frac{1}{\sqrt{2}} < x, \ \frac{1}{\sqrt{2-\delta}} < y \leq \sqrt{1-\varepsilon}.$$

Weak solutions in cases 2 and 3 are defined exactly analogously to case 1, with appropriate replacement of the domain and function spaces.

Theorem 1. *Let the lower-order terms in eqs. (6), (7) be distributed as in cases 1, 2, or 3, on the domains* Ω_1, Ω_2, *or* Ω_3, *respectively. Then there exists a weak solution of the boundary-value problem (5)-(7), (10) for every* $\mathbf{f} \in H$.

Proof. The proof is an extension of the arguments in Ref. 23, so we will be brief. We derive a *basic inequality*, that there is a $K \in \mathbb{R}^+$ such that $\forall \mathbf{w} \in W$,

$$K \left\| \mathbf{w} \right\|_* \leq \left\| L^* \mathbf{w} \right\|^*$$

(with the norms appropriately adjusted in cases 2 and 3). We derive this inequality by choosing a scalar multiplier a, computing the L^2 inner product $(L^*\mathbf{w}, a\mathbf{w})$, and integrating by parts. Denoting the coefficients of w_1^2 off the boundary by α, those of w_2^2 by γ and those of $w_1 w_2$ by 2β, we choose, in case 1, $a = x^2$ and obtain

$$\alpha = x\left(3x^2 - 1\right), \ \gamma = x\left(1 - y^2\right),$$

and

$$\beta = yx^2,$$

where

$$2\beta w_1 w_2 \geq -2x \left|x w_1\right| \left|y w_2\right| \geq -\left(x^3 w_1^2 + x y^2 w_2^2\right).$$

In case 2 we choose $a = 1$ and obtain

$$\alpha = 2x, \ \gamma = -2y,$$

and $\beta = 0$.

In case 3 we choose $a = xy$ and obtain

$$\alpha = \frac{y}{2}\left(3x^2 - 1\right), \ \gamma = \frac{y}{2}\left(1 - y^2\right),$$

and

$$2\beta = -\left(1 - y^2\right) x.$$

The quadratic form $\alpha\gamma - \beta^2$ can be shown to be non-negative in case 3 by noticing that the argument in Sec. 6.2 of Ref. 23 does not use the restriction $x \leq 1$ and thus extends to our more general case.

The remainder of the proof is essentially the same for all three cases. Applying Green's Theorem to derivatives of products in $(L^*\mathbf{w}, a\mathbf{w})$, we obtain a boundary integral I having the form

$$\int_{\partial\Omega} \frac{a}{2}\left[(1-x^2)w_1^2 dy + 2xyw_1^2 dx\right]$$

$$-\int_{\partial\Omega} a\left[(1-y^2)w_1 w_2 dx + \frac{1}{2}(1-y^2)w_2^2 dy\right].$$

Because w_1 vanishes identically on C, the boundary integral is nonnegative on C by the hypothesis on $dy_{|C}$. On the characteristic curves, we no longer have the property that $dx = 0$, which we used in deriving the basic inequality of Ref. 23. However,

$$I_{|\Gamma} = \int_{\Gamma} \frac{a}{2}\left\{(1-x^2)w_1^2 dy + [2xyw_1^2 - (1-y^2)w_1 w_2]dx\right\},$$

where we have used the fact that

$$w_2 dy = -w_1 dx$$

on characteristic lines. In fact, we have

$$I_{|\Gamma} = \int_{\Gamma} \frac{a}{2}\left[(1-x^2)w_1^2\left(\frac{dy}{dx}\right) + 2xyw_1^2 - (1-y^2)w_1 w_2\right] dx$$

$$= \int_{\Gamma} \frac{a}{2}\left[-(1-x^2)w_1 w_2\left(\frac{dy}{dx}\right)^2 + 2xyw_1^2 - (1-y^2)w_1 w_2\right] dx$$

by the same identity. Equation (1) implies that

$$-(1-x^2)\left(\frac{dy}{dx}\right)^2 = 2xy\frac{dy}{dx} + 1 - y^2,$$

so we can write

$$I = \int_\Gamma \frac{a}{2}\left[2xy\frac{dy}{dx} + 1 - y^2\right] w_1 w_2 dx +$$

$$\int_\Gamma \frac{a}{2}\left[2xyw_1(-w_2\frac{dy}{dx}) - (1-y^2)w_1 w_2\right] dx = 0.$$

This establishes the basic inequality.

Proceeding as in Ref. 19, we use the basic inequality to apply the Riesz Representation Theorem and obtain an element $\mathbf{h} \in H$ for which

$$-(\mathbf{w}, \mathbf{f}) = -(L^*\mathbf{w}, \mathbf{h})^*,$$

where the product on the right is the inner product on H (or on H' or L^2 in cases 2 or 3, respectively). Writing h_1 and h_2 of $\mathbf{h}$ in terms of appropriate rescalings of u_1 and u_2,[23] we obtain

$$-(L^*\mathbf{w}, \mathbf{h})^* = (L^*\mathbf{w}, \mathbf{u}),$$

which completes the proof.

2.2 Strong solutions

By a *strong solution* of the boundary-value problem (5), (10) we mean an element $\mathbf{u} \in L^2(\Omega)$ for which there exists a sequence $\mathbf{u}^\nu$ of continuously differentiable vectors satisfying the boundary condition (10), for which

$$\lim_{\nu\to\infty} \|\mathbf{u}^\nu - \mathbf{u}\|_{L^2} = 0,$$

and

$$\lim_{\nu\to\infty} \|L\mathbf{u}^\nu - \mathbf{f}\|_{L^2} = 0.$$

For $\mathbf{u} = (u_1(x,y), u_2(x,y))$, $(x,y) \in \Omega \subset\subset \mathbb{R}^2$, define the operator $L = (L_1, L_2)$ by the matrix equation

$$L\mathbf{u} = A^1\mathbf{u}_x + A^2\mathbf{u}_y + B\mathbf{u} \tag{13}$$

for matrices A^1, A^2, and B. We say that L is *symmetric-positive*[7,15,16] if the matrices A^1 and A^2 are symmetric and the matrix

$$Q \equiv 2B^* - A^1_x - A^2_y$$

is nonnegative. Here

$$B^* = \frac{1}{2}(B + B^t),$$

where for a matrix $W = [w_{ij}]$, $W^t = [w_{ji}]$.

In cases for which L is not symmetric-positive, there may be a nonsingular matrix E such that EL is symmetric-positive. In that case we replace the equation

$$L\mathbf{u} = \mathbf{f}$$

by the equation

$$EL\mathbf{u} = E\mathbf{f}$$

and try to show that the operator EL is symmetric-positive. (The conversion of L into a symmetric-positive operator by the construction of a suitable multiplier E will not be used in this section, but will be used in Sec. 3.3.)

Suppose that $N(x, y)$, $(x, y) \in \partial\Omega$, is a linear subspace of the vector space V, where $\mathbf{u}$ is regarded as a mapping $\mathbf{u} : \Omega \cup \partial\Omega \to V$, and that $N(x, y)$ depends smoothly on x and y. Define the matrix

$$\beta = n_1 A^1_{|\partial\Omega} + n_2 A^2_{|\partial\Omega},$$

where $\mathbf{n} = (n_1, n_2)$ is the outward-pointing normal vector to $\partial\Omega$. The boundary condition that u lie in N is said to be *admissible*[15] if N is a maximal subspace of V and if the quadratic form $(\mathbf{u}, \beta\mathbf{u})$ is non-negative on $\partial\Omega$.

A sufficient condition[7] for admissibility is that there exist a decomposition

$$\beta = \beta_+ + \beta_-,$$

for which the direct sum of the null spaces for β_+ and β_- spans the restriction of V to the boundary, the intersection of the ranges of β_+ and β_- have only the vector $\mathbf{u} = 0$ in common, and the matrix $\mu = \beta_+ - \beta_-$ satisfies

$$\mu^* = \frac{\mu + \mu^t}{2} \geq 0.$$

In this case the boundary condition

$$\beta_- \mathbf{u} = 0 \, on \, \partial\Omega$$

is admissible for the boundary-value problem

$$L\mathbf{u} = f \, in \, \Omega.$$

Moreover, the boundary condition

$$\beta_+^t \mathbf{w} = 0 \, on \, \partial\Omega$$

is admissible for the adjoint problem

$$L^* \mathbf{w} = \mathbf{h} \, in \, \Omega.$$

These two problems possess unique, strong solutions whenever the differential operators are symmetric-positive and the boundary conditions are admissible.[7,15]

In this section we give sufficient conditions for the existence of certain strong solutions arising from an arbitrarily small lower-order perturbation of the Laplace-Beltrami equations on extended $\mathbb{P}^2$. We do so by showing that the differential operator L given by (5)-(7) with $k_1 = k_2 = k_3 = k_4 = 0$ is arbitrarily close to a symmetric-positive operator and by stating an admissible boundary condition. The existence of strong solutions to a different perturbation on an explicit domain will be shown in Sec. 3.3.

If the matrices A^1 and A^2 of eq. (13) are given by eqs. (8) and (9) and the matrix B is given by

$$\begin{pmatrix} -2x & -2y \\ 0 & 0 \end{pmatrix},$$

then the quantity Q is zero. Thus we replace the matrix B by a matrix B_ε which differs from B by an arbitrarily small perturbation and takes the form

$$B_\varepsilon = \begin{pmatrix} -2x + \varepsilon_1 & -2y + \varepsilon_2 \\ (1 - y^2)\, \varepsilon_3 & (1 - y^2)\, \varepsilon_4 \end{pmatrix}, \tag{14}$$

where $\varepsilon_1 > 0$, $\varepsilon_4 > 0$, $\varepsilon_2 + (1 - y^2)\, \varepsilon_3 \geq 0$, and

$$\left[\varepsilon_2 + \left(1 - y^2\right) \varepsilon_3\right]^2 \leq 4 \left(1 - y^2\right) \varepsilon_1 \varepsilon_4.$$

If we choose the domain of L in such a way that $y^2 < 1$ there, then this replacement converts Q into a positive-definite matrix and L into a symmetric-positive operator.

Denote by Ω_4 a domain having C^2 boundary $\partial\Omega_4 = E \cup F$ such that $y^2 < 1$ on Ω_4. Let the components of the normal vector $\mathbf{n}$ on $\partial\Omega_4$ be given by (n_1, n_2). Assume that n_1 and n_2 never vanish at the same point of $\partial\Omega_4$. We place conditions on n_1, n_2, and $\partial\Omega_4$ sufficient to guarantee admissibility of the boundary condition

$$u_1 n_2 - u_2 n_1 = 0 \tag{15}$$

on F, with no condition given on E.

Let $n_1 \geq 0$, $n_2 \leq 0$ on F and $n_1 \leq 0$, $n_2 \geq 0$ on E. Defining the adjoint space as in Sec. 2.1, for $\mathbf{w} \in V^*$ we take $\mathbf{w} = (0, w_2)$ on F and

$$w_1 n_2 - w_2 n_1 = 0$$

on E. Define

$$\alpha = \left[-\left(1 - y^2\right) n_2/n_1 + 2xy - \left(1 - x^2\right) n_1/n_2 \right] n_2.$$

Assume that $\alpha = 0$ on E, and that $\alpha \leq 0$ on F.

Theorem 2. *The boundary-value problem*

$$L\mathbf{u} = A^1 \mathbf{u}_x + A^2 \mathbf{u}_y + B_\varepsilon \mathbf{u} = \mathbf{f}$$

for $(x, y) \in \Omega_4$, with A^1, A^2, and B_ε given by eqs. (8), (9), and (14) respectively and with condition (15) imposed on the curve F of $\partial\Omega_4$, possesses a unique, strong solution $\mathbf{u}(x, y)$ for every $\mathbf{f} \in L^2(\Omega_4)$.

Proof. Because the matrix B_ε has been constructed in such a way that L is symmetric-positive, it remains only to show that the boundary condition (15) is admissible on Ω_4.

We have

$$\beta = \begin{pmatrix} -\alpha - (1 - y^2)\, n_2^2/n_1 & (1 - y^2)\, n_2 \\ (1 - y^2)\, n_2 & -(1 - y^2)\, n_1 \end{pmatrix}.$$

Note that the apparent singularities in β at $n_1 = 0$ and in α at $n_2 = 0$ are removable.

On F, choose

$$\beta_+ = \begin{pmatrix} -\alpha & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\beta_- = \left(1 - y^2\right) n_2 \begin{pmatrix} -n_2/n_1 & 1 \\ 1 & -n_1/n_2 \end{pmatrix}.$$

On E, choose

$$\beta_+ = \left(1 - y^2\right) n_2 \begin{pmatrix} -n_2/n_1 & 1 \\ 1 & -n_1/n_2 \end{pmatrix}$$

and

$$\beta_- = \begin{pmatrix} -\alpha & 0 \\ 0 & 0 \end{pmatrix}.$$

If $\mathbf{u} \in V_{|F}$, then (15) implies that $\beta_- \mathbf{u} = 0$. The properties of V^* imply that $\mathbf{w}^t \beta_+ = 0$ for $\mathbf{w} \in V_{|F}^*$. If $\mathbf{u} \in V_{|E}$, then $\beta_- \mathbf{u} = 0$ for all values of $\mathbf{u}$ and $\mathbf{w}^t \beta_+ = 0$ by the properties of V^* and α. So the direct sum of the null spaces of β_- and β_+ spans V on $\partial\Omega_4$. Moreover, the hypotheses guarantee that the ranges of β_- and β_+ have only the zero vector in their intersection. Finally,

$$\beta_+ - \beta_- = \mu^* \geq 0$$

on both E and F.

This completes the proof of Theorem 2.

3 An analogous problem from optics

Geometrical optics is a zero-wavelength approximation to classical wave mechanics in which the governing differential equations are replaced by the Euclidean geometry of rays. The limitations of the geometrical optics approximation are apparent in the neighborhood of *caustics*, which are envelopes of a family of rays. It is not simply that geometrical optics predicts infinite intensity in such regions, whereas diffractive effects reduce the predicted intensity to a finite number. Even in applications for which the agreement between the predictions of geometrical optics and experiment is generally good, the former may predict singularities, *e.g.,* cusps, which are entirely smoothed out by diffraction. A dramatic example of this for the case of water waves is illustrated in Figures 5.6.1 and 5.6.2 of Ref. 29. This is, of course, far

from the only drawback of the geometrical optics approximation. See, for example, the discussion of the rainbow caustic in Sec. 6.3 of Ref. 22.

The accuracy of the geometrical optics approximation can be improved by considering waves of arbitrarily high frequency obtained by uniform asymptotic approximation of solutions to the Helmholtz equation (Sec. 3.1). While the older of these approximations also fail at caustics, an asymptotic formula introduced independently by Kravtsov[12] and Ludwig[17] retains its meaning even in the neighborhood of a caustic; see Ref. 13 for a review.

Recently, Magnanini and Talenti studied a nonlinear elliptic-hyperbolic equation, implied by the Ludwig-Kravtsov approximation, having the form[18]

$$\left(|\nabla v|^4 - v_y^2\right) v_{xx} + 2v_x v_y v_{xy} + \left(|\nabla v|^4 - v_x^2\right) v_{yy} = 0, \tag{16}$$

where $v = v(x, y)$, $(x, y) \in \mathbb{R}^2$. Those authors were able to show the existence of weak solutions to the full Dirichlet problem for the linear elliptic-hyperbolic equation

$$\left[\left(p^2 + q^2\right)^2 - p^2\right] V_{pp} - 2pqV_{pq} + \left[\left(p^2 + q^2\right)^2 - q^2\right] V_{qq} = 0, \tag{17}$$

which is related to eq. (16) by the *Legendre transformation*

$$V_L(p, q) = xp + yq - v(x, y). \tag{18}$$

Magnanini and Talenti's result is remarkable in that it is difficult to formulate a full Dirichlet problem which is well-posed for a given elliptic-hyperbolic equation, even in the weak sense; by *full* we mean that data are prescribed on the entire boundary. Morawetz's proof of the existence of weak solutions to the full Dirichlet problem for the Tricomi equation, the most intensively studied elliptic-hyperbolic equation, required a delicate argument.[20,27] The full Dirichlet problems for other important elliptic-hyperbolic equations remain unknown. For example, the full Dirichlet problem has not been correctly formulated even for weak solutions to a scalar elliptic-hyperbolic equation associated to electromagnetic wave propagation in cold plasma, although a well-posed Dirichlet problem for weak solutions has been formulated for data prescribed only on part of the boundary.[24] (In fact, Magnanini and Talenti do more than prove the existence of a weak solution: they also show uniqueness and internal regularity modulo a point, and give an explicit representation of the solution in terms of special functions.)

The existence of a well-posed Dirichlet problem is important because physical reasoning often suggests that the full Dirichlet problem is the correct problem even in the case of equations for which mathematical reasoning suggests otherwise.

Two questions suggested by Magnanini and Talenti's paper are:

i) The transformation (18) itself fails at caustics (which are not generally identical to the caustics of the physical model). One would like to characterize regions at which this linearization method fails and the nature of the singularities that arise in such regions. See, for example, Proposition 2 of Ref. 26.

ii) The result proven in Ref. 18 requires the domain boundary to lie entirely within the elliptic region of the equation. It is an important quality of eq. (17) that the elliptic region surrounds the hyperbolic region, a property not shared by other elliptic-hyperbolic equations. Thus there is some mathematical interest in asking whether solutions of (17) exist with boundary points lying in both the elliptic and hyperbolic regions, a situation in which this special condition is no longer applicable. We consider this question in Sec. 3.3.

Equation (16) is a special case of the system

$$\left[\left(p^2+q^2\right)^2-q^2\right]p_x+2pqp_y+\left[\left(p^2+q^2\right)^2-p^2\right]q_y=0, \tag{19}$$

$$p_y-q_x=0. \tag{20}$$

This system is equivalent to eq. (16) if there is a continuously differentiable scalar function $v\,(x,y)$ for which $v_x=p$ and $v_y=q$. (Such a function always exists locally, by eq. (20).)

Consider any two-dimensional quasilinear system of two equations having the form

$$\left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}\right]\frac{\partial}{\partial x}\left(\begin{array}{c} p \\ q \end{array}\right)+\left[\begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \end{array}\right]\frac{\partial}{\partial y}\left(\begin{array}{c} p \\ q \end{array}\right)=\left(\begin{array}{c} 0 \\ 0 \end{array}\right), \tag{21}$$

where the entries of the coefficient matrices depend only on p and q. Then the coordinate transformation $(x,y)\rightarrow(p,q)$ takes eq. (21) into the linear form

$$\left[\begin{array}{cc} b_{12} & -a_{12} \\ b_{22} & -a_{22} \end{array}\right]\frac{\partial}{\partial p}\left(\begin{array}{c} x \\ y \end{array}\right)+\left[\begin{array}{cc} -b_{11} & a_{11} \\ -b_{21} & a_{21} \end{array}\right]\frac{\partial}{\partial q}\left(\begin{array}{c} x \\ y \end{array}\right)=\left(\begin{array}{c} 0 \\ 0 \end{array}\right),$$

provided the Jacobian of the transformation

$$J = \frac{\partial\,(x, y)}{\partial\,(p, q)} = \frac{\partial x}{\partial p}\frac{\partial y}{\partial q} - \frac{\partial y}{\partial p}\frac{\partial x}{\partial q}$$

is nonzero. This special case of the Legendre transformation is called a *hodograph map*, and the space having coordinates (p, q) is called the *hodograph plane;* see, *e.g.,* Sec. V.2.2 of Ref. 5.

The coordinate systems (p, q) and (x, y) are related by eq. (18),where

$$(x, y) = \left(\frac{\partial V}{\partial p}, \frac{\partial V}{\partial q}\right)$$

and

$$(p, q) = \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right).$$

Applying a hodograph transformation to eqs. (19), (20) yields the system

$$\left[\left(p^2 + q^2\right)^2 - p^2\right] x_p - 2pqx_q + \left[\left(p^2 + q^2\right)^2 - q^2\right] y_q = 0, \tag{22}$$

$$x_q - y_p = 0. \tag{23}$$

This system is equivalent to eq. (17) if there is a continuously differentiable scalar function $V\,(x, y)$ for which $V_p = x$ and $V_q = y$. (Again, this can always be arranged locally.)

As in Sec. 2, we write the second-order terms of eqs. (22), (23) in the form $A^1\mathbf{u}_x + A^2\mathbf{u}_y$, where $\mathbf{u} = \mathbf{u}(x, y)$ and in this case

$$A^1 = \begin{bmatrix} \left(x^2 + y^2\right)^2 - x^2 & 0 \\ 0 & -1 \end{bmatrix}$$

and

$$A^2 = \begin{bmatrix} -2xy & \left(x^2 + y^2\right)^2 - y^2 \\ 1 & 0 \end{bmatrix}.$$

The characteristic equation

$$\left|A^1 - \lambda A^2\right| = -\left\{\left[\left(x^2 + y^2\right)^2 - y^2\right]\lambda^2 + 2xy\lambda + \left[\left(x^2 + y^2\right)^2 - x^2\right]\right\}$$

possesses two real roots λ_1, λ_2 precisely when $x^2 + y^2 > \left(x^2 + y^2\right)^2$, that is, when $x^2 + y^2 < 1$. Thus the system is hyperbolic at points lying inside the open unit disc centered at $(x, y) = (0, 0)$ and elliptic outside the closure of this disc. The circle $x^2 + y^2 = 1$, along which the change in type occurs, is the parabolic region of the system.

3.1 Uniform asymptotic approximations

Substitution of the simplest formula for an oscillatory wave into the wave equation results in the *Helmholtz equation*

$$\Delta U(\mathbf{x}) + k^2\nu^2 U(\mathbf{x}) = 0, \tag{24}$$

where we take $\mathbf{x}$ to be a vector in $\mathbb{R}^2$, and where k and ν are physical constants. In the standard application, ν is the refractive index of the medium and k is inversely proportional to wavelength. In the region of visible light, the wavelength is sufficiently small that k dominates over all other mathematically relevant parameters, an undesirable property known as *stiffness.*

For this reason, short-wave solutions of (24) are usually approximated by *uniform asymptotic expansions*[12,17] which satisfy (24) to arbitrarily high order in k^{-1}. These approximations are valid in regions which contain smooth and convex caustics such as a circular caustic. The size of the region of validity is independent of k. Take $\nu \equiv 1$ and approximate the solution to (24) by an expansion having the form

$$U_{approx}(x,y) =$$

$$\left\{ Z\left(k^{2/3}u\right)\left(\sum_{j=0}^{\infty} W_j(\mathbf{r})\cdot(ik)^{-j}\right) + \frac{i}{k^{1/3}} Z'\left(k^{2/3}u\right)\left(\sum_{j=0}^{\infty} X_j(\mathbf{r})\cdot(ik)^{-j}\right)\right\}$$

$$\times \exp\left[ikv(x,y)\right],$$

where $u(x,y)$, $v(x,y)$, $W_j(\mathbf{r})$, and $X_j(\mathbf{r})$ are functions which do not depend on k and which are to be determined with the solution; the function $Z(t)$ is a solution of the *Airy equation*

$$Z''(t) - tZ(t) = 0,$$

with initial conditions

$$Z(0) = \frac{3^{-2/3}}{\Gamma(2/3)}$$

and

$$Z'(0) = -\frac{3^{-1/3}}{\Gamma(1/3)},$$

where $\Gamma(\,)$ is the gamma function.

This model implies the following system of equations for u and v :

$$u\left(u_x^2+u_y^2\right)-\left(v_x^2+v_y^2\right)+1=0,$$

$$u_x v_x+u_y v_y=0.$$

In Ref. 18 three possible solutions of this system are enumerated:

$$u=0,\ |\nabla v|^2=1;$$

$$|\nabla u|=0,\ |\nabla v|^2=1;$$

the third possibility is that eq. (16) is satisfied.

Obviously, the third alternative is the most interesting, and this case is studied in Ref. 18. This case is linearized to eq. (17) by a hodograph transformation.

3.2 A first-order system

Thus we are led to a system resembling eqs. (5)-(7):

$$L\mathbf{u}=\mathbf{g}, \tag{25}$$

where

$$L=\left(L_1,L_2\right),\ \mathbf{g}=\left(g_1,g_2\right),$$

$$\mathbf{u}=\left(u_1\left(x,y\right),u_2\left(x,y\right)\right),\ \left(x,y\right)\in\Omega\subset\subset\mathbb{R}^2,$$

$$\left(L\mathbf{u}\right)_1=\left[f\left(x,y\right)-x^2\right]u_{1x}-2xyu_{1y}+\left[f\left(x,y\right)-y^2\right]u_{2y} \tag{26}$$

and

$$\left(L\mathbf{u}\right)_2=\left[f\left(x,y\right)-y^2\right]\left(u_{1y}-u_{2x}\right), \tag{27}$$

for

$$f\left(x,y\right)=\left(x^2+y^2\right)^2. \tag{28}$$

The domain is chosen so that

$$f\left(x,y\right)-y^2\neq 0,$$

under which system (25)-(28) becomes an inhomogeneous generalization of eqs. (22), (23). If in particular, $g_1=g_2=0$, $u_1=V_x$, and $u_2=V_y$, where $V\left(x,y\right)$ is a scalar function, then eqs. (25)-(28) reduce to eq. (17).

As in the preceding sections, the second-order terms of eqs. (25)-(28) can be written in the form $A^1\mathbf{u}_x + A^2\mathbf{u}_y$, where

$$A^1 = \begin{bmatrix} f(x,y) - x^2 & 0 \\ 0 & -\left(f(x,y) - y^2\right) \end{bmatrix}$$

and

$$A^2 = \begin{bmatrix} -2xy & f(x,y) - y^2 \\ f(x,y) - y^2 & 0 \end{bmatrix}.$$

We find that the system is hyperbolic in the intersection of Ω with the open unit disc centered at $(0,0)$ and elliptic in the intersection of Ω with the complement of the closure of this disc.

3.3 Strong solutions in an annulus

Writing eq. (17) in polar coordinates (r,θ), $r \geq 0$, $0 < \theta \leq 2\pi$, we obtain[18]

$$\left(r^2 - 1\right) V_{rr} + rV_r + V_{\theta\theta} = 0. \tag{29}$$

Letting $u_1 = V_r$ and $u_2 = V_\theta$ transforms eq. (29) into a first-order system of the form

$$L\mathbf{u} = A^1\mathbf{u}_r + A^2\mathbf{u}_\theta + B\mathbf{u} = \mathbf{f}, \tag{30}$$

with $\mathbf{u} = \left(u_1(r,\theta), u_2(r,\theta)\right)$, $\mathbf{f} = (0,0)$,

$$A^1 = \begin{pmatrix} r^2 - 1 & 0 \\ 0 & -1 \end{pmatrix},\ A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{31}$$

and

$$B = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}.$$

As in Sec. 2.2, the matrices are symmetric and we find that $Q = 2B^* - A^1_r - A^2_\theta$ is exactly zero, suggesting that an arbitrarily small perturbation of the matrix B will result in a symmetric-positive operator. However, we find that we can retain the consistency condition $u_{1\theta} - u_{2r} = 0$ if we employ a multiplier E as described in Sec. 2.2. Thus we define

$$E = \begin{pmatrix} a & c\left(1 - r^2\right) \\ c & a \end{pmatrix},$$

where $a = a(r,\theta)$ and $c = c(r,\theta)$ are continuously differentiable functions to be chosen. We replace B by the matrix

$$B_\varepsilon = \begin{pmatrix} r+\varepsilon_1 & \varepsilon_2 \\ 0 & 0 \end{pmatrix}, \tag{32}$$

where $\varepsilon_1, \varepsilon_2$ are arbitrarily small, positive constants.

Replacing eq. (30) by the system

$$EL = EA^1\mathbf{u}_r + EA^2\mathbf{u}_\theta + EB_\varepsilon\mathbf{u} = E\mathbf{f}, \tag{33}$$

with A^1, A^2, and B_ε given by eqs. (31) and (32), we find that EL is a symmetric-positive operator provided we choose $0 \leq \varepsilon_0 \leq r \leq R < \infty$, c a positive constant, and

$$a = Me^{\varepsilon_2\theta} + \frac{\left(\sqrt{2}-\varepsilon_1\right)c}{\varepsilon_2},$$

where M is a constant such that $M >> c$.

We will solve eqs. (33) in the annulus Ω_5 given by $\varepsilon_0 \leq r \leq \sqrt{2}$. (The solutions can be patched into an elliptic boundary-value problem on the annulus $\sqrt{2} \leq r \leq R$.) Data will be prescribed on the outer boundary only. Annular domains are natural when numerical methods are used to study an equation, such as eq. (17), which is known to be singular at the origin, with the singular point excluded. The problem is also of some historical interest. An equation differing from (17) only in its lower-order terms was one of the first elliptic-hyperbolic equations to be studied, more than 75 years ago, by Bateman (Sec. 9 of Ref. 1). That equation equation arose from the solution of Laplace's equation in toroidal coordinates.[2] At the time, Bateman raised the question of the existence and uniqueness of solutions in an annular region containing the unit circle, in which the outer boundary lies in the elliptic region and the inner boundary lies in the hyperbolic region of the equation.

Although the system that we consider is a small perturbation of the one studied in Ref. 18, we note that the original equation is itself an approximation, as described in Sec. 3.1.

Theorem 3. *Equations (33) with boundary conditions*

$$\tau(\theta)u_1 + \sigma(\theta)u_2 = 0,\ \sigma^2(\theta) > \tau^2(\theta) \tag{34}$$

imposed on the outer boundary $r = \sqrt{2}$, *possess a strong solution on the annulus* Ω_5 *for every* $\boldsymbol{f} \in L^2(\Omega_5)$.

Proof. Although the equations are different, the argument is similar to the proof by Torre[30] of the corresponding assertion for the helically reduced wave equation.

The matrices E and B_ε have been constructed in such a way that the operator EL is manifestly symmetric-positive (for large M), and the proof again reduces to a demonstration that the boundary conditions are admissible. At the outer boundary, choose $\mathbf{n}_{outer} = dr$. Then

$$\beta_{outer} = \begin{pmatrix} a & c \\ c & -a \end{pmatrix}.$$

Choose

$$\beta_{outer^-} = \frac{1}{\sigma^2+\tau^2}\begin{pmatrix} \sigma\tau c + \tau^2 a & \sigma^2 c + \sigma\tau a \\ -\sigma\tau a + \tau^2 c & -\sigma^2 a + \sigma\tau c \end{pmatrix}.$$

Then

$$\beta_{outer^+} = \frac{1}{\sigma^2+\tau^2}\begin{pmatrix} -\sigma\tau c + \sigma^2 a & \tau^2 c - \sigma\tau a \\ \sigma\tau a + \sigma^2 c & -\tau^2 a - \sigma\tau c \end{pmatrix}.$$

Notice that $\beta_{outer^+} + \beta_{outer^-} = \beta_{outer}$ and that $\beta_{outer^-}\mathbf{u} = 0$, as (34) implies that $u_2 = -(\tau/\sigma)u_1$ on the circle $r = \sqrt{2}$. Moreover,

$$\mu = \frac{1}{\sigma^2+\tau^2}\begin{pmatrix} (\sigma^2-\tau^2)\,a - 2\sigma\tau c & (\tau^2-\sigma^2)\,c - 2\sigma\tau a \\ (\sigma^2-\tau^2)\,c + 2\sigma\tau a & (\sigma^2-\tau^2)\,a - 2\sigma\tau c \end{pmatrix},$$

implying that

$$\mu^* = \frac{1}{\sigma^2+\tau^2}\begin{pmatrix} (\sigma^2-\tau^2)\,a - 2\sigma\tau c & 0 \\ 0 & (\sigma^2-\tau^2)\,a - 2\sigma\tau c \end{pmatrix}.$$

But this matrix is non-negative, given that $\sigma^2 > \tau^2$, provided that we choose M sufficiently large.

On the inner boundary we choose

$$\mathbf{n}_{inner} = \left(\varepsilon_0^2 - 1\right)^{-1} dr.$$

Then

$$\beta_{inner} = \begin{pmatrix} a & c \\ c & -\left(\varepsilon_0^2 - 1\right)^{-1} a \end{pmatrix}.$$

Choose

$$\beta_{inner^-} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $\beta_{inner^+} = \beta_{inner}$ and $\mu^* \geq 0$ for M sufficiently large. Moreover, $\beta_{inner^-}\mathbf{u} = 0$ on the circle $r = \varepsilon_0$ for any vector $\mathbf{u}$.

This completes the proof of Theorem 3.

4 A remark on terminology and notation

Hodge[8] originally considered a p-form ω to be harmonic if it satisfies the first-order equations

$$d\omega = \delta\omega = 0, \tag{35}$$

where $d : \Lambda^p \to \Lambda^{p+1}$ is the exterior derivative and $\delta : \Lambda^{p+1} \to \Lambda^p$ is the adjoint of d. If the underlying space is $\mathbb{R}^2$ and ω is a 1-form given by

$$\omega = pdx + qdy,$$

where p and q are continuously differentiable functions, then the Hodge equations (35) reduce to the Cauchy-Riemann equations for p and $-q$. However, although d is independent of the underlying metric, its adjoint δ has a different local form for different metrics. Thus for a surface having metric tensor g_{ij}, the Hodge equations for 1-forms are equivalent to the system (3), (4). A discussion of exterior forms and their properties is given in, *e.g.,* Ref. 21.

The standard definition of a *harmonic form* is given in terms of a second-order operator: it is a solution of the form-valued Laplace-Beltrami equations

$$(d\delta + \delta d)\,\omega = 0.$$

If the domain has *zero boundary* (either no boundary or the prescribed value $\omega \equiv 0$ on the boundary), then the definitions in terms of first- and second-order operators are equivalent. Otherwise, one distinguishes them by calling

a form that satisfies eqs. (35) a *harmonic field*. In words, the Hodge equations assert that a harmonic field ω is both *closed* ($d\omega = 0$) and *co-closed* ($\delta\omega = 0$) under the exterior derivative d. Obviously, every harmonic field is a harmonic form, but the converse is false.

Notice that in eqs. (6) and (7), $L_1 \neq \delta$ and $L_2 \neq d$. For example, L_2 includes a factor of $1 - y^2$ whereas d does not, and δ includes determinants of the metric tensor, whereas L_1 does not. In addition, cases 1 and 2 of (6), (7) include additional lower-order terms. Thus for example δ and d are self-adjoint, whereas L_1 and L_2 are not unless $k_1 = k_2 = k_3 = k_4 = 0$.

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